Variational solutions of Hamilton-Jacobi equations - 1 Prologue

INDAM - Cortona, Il Palazzone

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At the beginning

• Classical Hamilton-Jacobi equations arose, maybe first, inside the framework of the **celestial mechanics** and early **canonical transformations** theory: Lagrange, Hamilton, Jacobi, Poisson, Poincaré, Weierstrass,...

• .... to solve the Hamiltonian systems of ODE *by means of* a *suitable* solution (a *complete integral*) of a H-J equation, a PDE

• afterwards, H-J equations became central into the study of **wave propagation** ... *in an inverse* direction:

to solve a PDE equation (H-J for wave, e.g. eikonal), *by means of* solutions (*characteristic curves*), a Hamiltonian systems of ODE

Weak solutions

• Hopf, Kružkov, Benton,
and few others,
were the true pioneers towards global week solutions for H-J equations

• at the end, in the early 80's, Crandall-Evans-Lions drew **viscosity solutions** theory

• In this new environment, the 1965 weak proposal by Hopf is shown to be, precisely, the viscous solution of an initial Cauchy problem [Bardi and Evans]

Weak KAM

and finally, here
• **Weak KAM** theory
Lions-Papanicolaou-Varadhan, Fathi,. . .

a sort of closure of the above order of ideas:

• a powerful effort to come back the original issue:
solve, even though in a weak form, a **stationary H-J**,
crucial
for the flow problems of analytical mechanics

- Come back a little to Hopf
- here the exemplary review of the Hopf' paper by Oleinik :

MR0182790 (32 #272) 35.35 (35.20)
Hopf, Eberhard
Generalized solutions of non-linear equations of first order.
J. Math. Mech. **14** 1965 951–973

A method is given for construction of generalized solutions of the Cauchy problem with initial data for $t = 0$ for nonlinear first-order equations of the type $(*)$ $z_t + f(z_x) = 0$ in several independent variables, where $z_x = (z_{x_1}, \cdots, z_{x_n})$. For these generalized solutions the Lipschitz condition is fulfilled, and they satisfy equation $(*)$ almost everywhere. The method given here generalizes the classical Jacobi theory of enveloping families of solutions of equations of the first order. An explicit formula for the solution of Cauchy's problem for equation $(*)$ with the condition $z|_{t=0} = z_0$ is obtained in the case $f'' \neq 0$, and also in the case of arbitrary continuous $f(u)$ and convex initial function z_0. This formula has the aspect:

$$z(t, x) = \inf_y \sup_u [z^o(y) + u \cdot (x - y) - f(u)t].$$

Stability in norm C is proved for the constructed generalized solutions of the Cauchy problem for variation of the initial function. Generalized solutions of the Cauchy problem for equation $(*)$ are also considered in the work of S. N. Kružkov [Dokl. Akad. Nauk SSSR **155** (1964), 743–746; MR0164137 (29 #1436)].

Reviewed by *O. A. Oleĭnik*

Hopf

• Some aspects to point out from the above 1965 paper by Hopf:

• The H-J equation is evolutive: $\frac{\partial z}{\partial t} + f(\frac{\partial z}{\partial x}) = 0$, and the Hamiltonian $f(u)$ is **convex**

• there is an 'enveloping' procedure –like Huygens Principle– of global **Generating Functions**. G. F. : suitable families of classical solutions of the H-J equation

• and the final outcome of the above enveloping procedure gives us a weak **Lipschitz** solution $z(t, x)$

Complete Integrals as Generating Functions

- $S(t,x,a) := -f(a)t + x \cdot a$, Complete Integral, is a solution of
- $\frac{\partial z}{\partial t} + f(\frac{\partial z}{\partial x}) = 0$ for any $a \in \mathbb{R}$
- so that
- $$W(t,x,y;a) := S(t,x,a) - S(t,y,a)$$
that is
- $$W(t,x,y;a) = -f(a)t + (x-y) \cdot a$$
- becomes a sort of **Green (geometrical) propagator**,

- $$z(t,x;y,a) = W(t,x,y;a) + z^0(y),$$ (prototype of **Generating Function)**,
- y: initial point x: final point
- y and a: auxiliary parameters, to remove by enveloping inf/sup procedure, obtaining, at the end, the Lipschitz solution $z(t,x)$

What we learn from it?

• After Hopf, the use of generating functions has not been fully *friendly* inside viscosity environment,
nevertheless,
• Bardi, Capuzzo-Dolcetta, Faggian, Osher,... : important constructions,

• but no general viscosity-like theory based on generating functions is still known

A road map

• A systematic search on H-J by Generating Functions arose in '80

• First, in a merely geometric context: Tulczyjew, Benenti, Libermann, Marle...

• Then, in a more topological and variational environment: Chaperon, Laudenbach, Sikorav, Viterbo...

A road map

• We will see that the notion of **Generating Function** is strictly connected to the **Lagrangian submanifolds of the symplectic manifolds**,

• so, we will begin from the description of the symplectic environment

A road map

- A systematic search on H-J by Generating Functions arose in '80

Variational solutions of Hamilton-Jacobi equations -2 geometrical setting: the symplectic environment

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Background: Symplectic manifolds

- “The name **complex group** formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word **complex** in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective **symplectic**.”

  Hermann Weyl, The classical Groups.

- Alan Weinstein's warning:

- *symplectic* is also the name for a bone in the head of the Teleòstei (fishes)

Symplectic manifolds

- (P, ω) is a **symplectic manifold** if :

P is a manifold,
- $\omega \in \Omega^2(P)$ is a 2-form, **closed** $(d\omega = 0)$ and **non degenerate**
- This is forcing: $\dim P = 2n$

- Main example, cotangent bundles:
- $P = T^*M$, where M is a n-dim (base) manifold
- in this case: $\omega = d\vartheta$ where ϑ is the Liouville 1-form,
- $\vartheta = \sum_i p_i dq^i \qquad \vartheta \in \Omega^1(T^*M)$
- $\omega = d\vartheta = \sum_i dp_i \wedge dq^i \qquad$ or : $\omega = \sum_i (dp_i \otimes dq^i - dq^i \otimes dp_i)$

- Rem: 'more general' 1-forms on T^*M are like:

$\bar{\theta} = \sum_i A_i(q,p) dq^i + \sum_j B^j(q,p) dp_j$

Lagrangian submanifolds

- An embedding $j:\Lambda\to T^*M$ is called *Lagrangian* if
- (i) Λ is of dimension $n=\frac{\dim T^*M}{2}=\dim M$
- (ii) $j^*\omega=0$ (or: $\omega|_\Lambda=0$)

- $j\left(\Lambda\right)$ is said a Lagrangian submanifold of (T^*M,ω)

- (ii) $\Rightarrow 0=j^*d\vartheta=dj^*\vartheta$ that is
- $j^*\vartheta$ (or: $\vartheta|_\Lambda$) is *closed*

- An embedding $j:\Lambda\to T^*M$ such that the pull-back $j^*\vartheta$ is *exact*
- is called *exact Lagrangian embedding* into T^*M

- Example:

The image $im(\mu)$ of a **closed 1-form** $\mu: \; M \to T^*M, \;\; d\mu = 0,$

$\mu: \; M \to T^*M, \quad d\mu = \sum_{ij} \frac{\partial \mu_i}{\partial q_j} dq^j \wedge dq^i = 0$

is a **Lagrangian** submanifold

- in fact:
-
- (i) $\dim(im(\mu)) = n$
- (ii) $\omega|_{im(\mu)} = \sum_i dp_i \wedge dq^i|_{im(\mu)} = \sum_{ij} \underbrace{\frac{\partial \mu_i}{\partial q_j}}_{sym} \underbrace{dq^j \wedge dq^i}_{skw} = 0$

- In particular, for any $f : M \to \mathbb{R}$, $\quad$ (df is an exact 1-form)

-

$$im(df) \text{ is a Lagrangian submanifold}$$

Parameterization of the Lagrangian submanifolds 2

- Conversely, giving a Lagrangian $\Lambda \subset T^*M$,

$$\begin{array}{ccccc} & j & & \pi_M & \\ \Lambda & \hookrightarrow & T^*M & \longrightarrow & M \\ \lambda & \mapsto & (q(\lambda), p(\lambda)) & \mapsto & q(\lambda), \end{array}$$

in case $\pi_M \circ j$ is locally invertible,

$$\operatorname{rk} D(\pi_M \circ j)(\lambda) = n = \max$$

- by [inverse function th.],

$$M \ni q \mapsto \bar{\lambda}(q) \in \Lambda$$

- by [Poincaré lemma],

$$\Lambda \ni \lambda \mapsto \bar{f}(\lambda) \in \mathbb{R}$$

is a local primitive of the closed $j^*\vartheta$, $\quad d\bar{f} = j^*\vartheta$,
- then $\Lambda =_{\text{locally}} im(df)$ for $f(q) := \bar{f} \circ \bar{\lambda}(q)$

- In other words; any Lagrangian submanifold, locally **transverse** to the fibers of the projection $\pi_M : T^*M \to M$, is parameterized by some suitable (local) real valued function f.

- Back to the general setting,

$$\begin{array}{ccccc} & j & & \pi_M & \\ \Lambda & \hookrightarrow & T^*M & \longrightarrow & M \\ \lambda & \mapsto & (q(\lambda), p(\lambda)) & \mapsto & q(\lambda), \end{array}$$

- what's happening when

$$\operatorname{rk} D(\pi_M \circ j)(\lambda) < n \quad ?$$

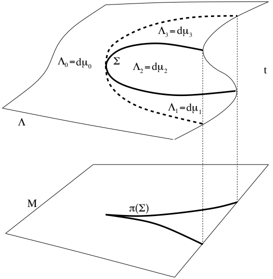
- Λ is 'multivalued', like a Riemann surface in Complex Analysis,

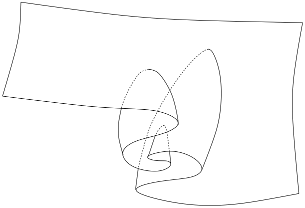
- $\Rightarrow$ **Maslov-Hörmander Theorem** (it is a local theorem):

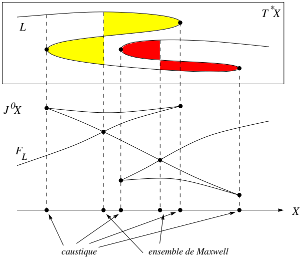
Maslov V.P., *Perturbation theory and asymptotic methods*, Moscow, 1965

Hörmander L., *Fourier integral operators I*, Acta Math., 1971

Examples of Lagrangian submanifolds - pictures







• **Maslov-Hörmander Theorem** : In any situation with respect **transversality** to the fibers of π_M, locally, Lagrangian submanifolds are necessarily descripted by **Generating Families**[1] $W(q,u)$:

$$M \times \mathbb{R}^k \ni (q^i, u^A) \longmapsto W(q^i, u^A) \in \mathbb{R}$$

• in the following way:

$$\Lambda = \{(q,p): \quad p_i = \frac{\partial W}{\partial q^i}(q^i, u^A), \quad 0 = \frac{\partial W}{\partial u^B}(q^i, u^A)\} \qquad (*)$$

• Furthermore, **zero** in $\mathbb{R}^k$ is a **regular value** for the map $Q \times \mathbb{R}^k \ni (q,u) \mapsto \frac{\partial W}{\partial u} \in \mathbb{R}^k$, that is

$$\mathrm{rk}\left(\frac{\partial^2 W}{\partial u^A \partial q^i} \quad \frac{\partial^2 W}{\partial u^A \partial u^B}\right)\Big|_{(*)} = k(= \mathrm{maximal}). \qquad (**)$$

[1] Sometimes said **Morse Families**

- $u = (u_A)_{A=1,\dots,k} \in \mathbb{R}^k$: **auxiliary parameters**

- In the case of transversality, we can choose $k = 0$, so $W = W(q)$

- In general, we have to choose: $k \geq \dim M - \mathrm{rk}[D(\pi_M \circ j)(\lambda_0)]$

$\Rightarrow$ We cannot involve a number of aux. parameters smaller than the loss of the rank of $D(\pi_M \circ j)(\lambda_0)$

• The above description of the **Generating Families** is unique up to the following **three** operations[2] [3]:

• 1. *Addition of constant:*
$\overline{W}(q;u) = W(q;u) + const. \approx W(q;u) \quad$ (trivial)

• 2. *Stabilization (i.e., addition of n.deg. quadratic forms):*
$\overline{W}(q;u,v) = W(q;u) + v^T A v \quad \approx \quad W(q;u) \quad v \in \mathbb{R}^{\bar{k}},\ \forall \det A \neq 0$

• 3. *Fibered diffeomorphism:*
For any fibered diffeomorphism

$$M \times \mathbb{R}^k \longrightarrow M \times \mathbb{R}^k$$

$$(q,v) \longmapsto (q, \bar{u}(q,v))$$

$$\overline{W}(q;v) := W(q;\bar{u}(q,v)) \quad \approx \quad W(q;u)$$

[2] A. Weinstein, *Lectures on symplectic manifolds*, 1976

[3] P. Libermann, C.-M. Marle, *Symplectic geometry and analytical mechanics*, 1987

Checking the invariance under fibered diffeomorphisms (Operation 3.):

$$M \times \mathbb{R}^k \longrightarrow M \times \mathbb{R}^k$$

$$(q, v) \longmapsto (q, \bar{u}(q, v))$$

$$\overline{W}(q; v) := W(q; \bar{u}(q, v))$$

$$\begin{aligned}
\overline{\Lambda} &= \Big\{(q,p) : \quad p = \tfrac{\partial \overline{W}}{\partial q}(q;v), \quad 0 = \tfrac{\partial \overline{W}}{\partial v}(q;v)\Big\} \\
&= \Big\{(q,p) : \quad p = \tfrac{\partial W}{\partial q} + \tfrac{\partial W}{\partial u}\tfrac{\partial \bar{u}}{\partial q}, \quad 0 = \tfrac{\partial W}{\partial u} \underbrace{\frac{\partial \bar{u}}{\partial v}}_{\det \neq 0}\Big\} \\
&= \Big\{(q,p) : \quad p = \tfrac{\partial W}{\partial q}, \quad 0 = \tfrac{\partial W}{\partial u}\Big\} \\
&= \Lambda \qquad \Rightarrow \qquad \overline{W}(q;v) \quad \approx \quad W(q;u)
\end{aligned}$$

$\square$

Maslov-Hörmander Theorem - Full reduction of parameters

- By Operation 2., i.e., Stabilization by adding quadratic forms, the number of aux. parameters can **increase**

- The number of aux. parameters can also **decrease:**

Whenever the max rank of

$$\mathrm{rk}\left(\frac{\partial^2 W}{\partial u^A \partial q^i} \quad \frac{\partial^2 W}{\partial u^A \partial u^B}\right)\Bigg|_{\frac{\partial W}{\partial u^A}=0} = k(=\mathrm{maximal})$$

can be detected from $k \times k$-matrix : $\qquad \det \frac{\partial^2 W}{\partial u^A \partial u^B}\Big|_{\frac{\partial W}{\partial u^A}=0} \neq 0$

it is possible, locally, fully to remove all the aux. par.; by implicit function th.,

$$\frac{\partial W}{\partial u^A} = 0 \Rightarrow \quad u^A = \hat{u}^A(q)$$

- so

$$\hat{W}(q) := W(q, \hat{u}^A(q))$$

is a Generating Function for the **same** Lagrangian submanifold:

$$p = \frac{\partial \hat{W}}{\partial q} = \frac{\partial W}{\partial q}(q, \hat{u}(q)) + \underbrace{\frac{\partial W}{\partial u}(q, \hat{u}(q))}_{=0} \frac{\partial \hat{u}}{\partial q}$$

- By Operation 2., i.e., Stabilization by adding quadratic forms, the number of aux. parameters can **increase**

- The number of aux. parameters can also **decrease:**

Whenever from the $k \times k$-matrix $\frac{\partial^2 W}{\partial u^A \partial u^B}$

$$\mathrm{rk}\left(\frac{\partial^2 W}{\partial u^A \partial q^i} \quad \frac{\partial^2 W}{\partial u^A \partial u^B}\right)\Big|_{\frac{\partial W}{\partial u^A}=0} = k (= \text{maximal})$$

it is possible to detect some (smaller) non-degenerate $h \times h$-sub-matrix, $h \leq k$,

$$\det \frac{\partial^2 W}{\partial u^\alpha \partial u^\beta} \neq 0, \qquad \alpha, \beta = 1, \ldots, h \leq k$$

then, by implicit function th.,

$$\frac{\partial W}{\partial u^\alpha}(q, u^\alpha, u^\Gamma) = 0 \quad \Rightarrow \quad u^\alpha = \hat{u}^\alpha(q, u^\Gamma), \quad \Gamma = h+1, \ldots, k$$

-
$$\hat{W}(q, u^\Gamma) := W(q, \hat{u}^\alpha(q, u^\Gamma))$$

is a Generating Function for the **same** Lagrangian submanifold.

Canonical Transformations

- Consider two manifolds, or two copies of a same manifold Q:
- Q_1 and Q_2
- T^*Q_1 and T^*Q_2
- (T^*Q_1, ω_1) and (T^*Q_2, ω_2)
- Diffeomorphisms

$$f : T^*Q_1 \longrightarrow T^*Q_2$$

$$(\bar{q}, \bar{p}) \longmapsto f(\bar{q}, \bar{p}) = (q, p)$$

preserving the respective **symplectic structures**, that is,

- $$\omega_1 = f^*\omega_2$$

- are said Canonical Transformations or Symplectomorphisms
- Main example: At any fixed time $t \in \mathbb{R}$, flows of Hamilton ode systems $\phi^t_{X_H}$ are Canonical Transformations:

$$\frac{d}{dt}\phi^t_{X_H} = X_H(\phi^t_{X_H})$$

Other Symplectic Manifolds: a frame for Canonical Transformations

- Consider the following graph-structure: $P := T^*Q_1 \times T^*Q_2$

with projections:

$$T^*Q_1 \xleftarrow{PR_1} T^*Q_1 \times T^*Q_2 \xrightarrow{PR_2} T^*Q_2$$

- Equip $T^*Q_1 \times T^*Q_2$ with the closed 2-form

$$\Omega := PR_2^*\omega_2 - PR_1^*\omega_1$$

- It turns out that $(T^*Q_1 \times T^*Q_2, \Omega)$ is a symplectic manifold

- **Theorem** A diffeomorphism $f : T^*Q_1 \to T^*Q_2$ is Canonical iff

$$\Lambda := \mathrm{graph}\,(f) \subset T^*Q_1 \times T^*Q_2$$

is a Lagrangian submanifold of the symplectic manifold $(T^*Q_1 \times T^*Q_2, \Omega)$.

- In fact:

$$0 = \Omega|_{\mathrm{graph}\,(f)} = f^*(\omega_2) - \omega_1$$

$\square$

$T^*Q_1 \times T^*Q_2$ is isomorphic to $T^*(Q_1 \times Q_2)$

- Observe that $T^*Q_1 \times T^*Q_2$ is **isomorphic** in a natural way to $T^*(Q_1 \times Q_2)$,

$$
\begin{tikzcd}
 & T^*Q_1 \times T^*Q_2 \arrow[dl, "PR_1"'] \arrow[dr, "PR_2"] & \\
T^*Q_1 & T^*(Q_1 \times Q_2) \arrow[u, "\varphi"'] & T^*Q_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
TQ_1 \arrow[d, "\tau_{Q_1}"'] & T(Q_1 \times Q_2) \arrow[l, "Tpr_1"'] \arrow[r, "Tpr_2"] \arrow[d, "\tau_{Q_1 \times Q_2}"'] & TQ_2 \arrow[d, "\tau_{Q_2}"] \\
Q_1 & Q_1 \times Q_2 \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & Q_2
\end{tikzcd}
$$

in local charts:

$$\varphi(q_{(1)}, q_{(2)}; p_{(1)}, p_{(2)}) = (q_{(1)}, p_{(1)}; q_{(2)}, p_{(2)}).$$

Maslov-Hörmander in $(T^*Q_1 \times T^*Q_2, \Omega)$

• We recall that the setting of Maslov-Hörmander's th. in T^*Q was laid down on

$$\begin{array}{ccccc} & j & & \pi_Q & \\ \Lambda & \hookrightarrow & T^*Q & \longrightarrow & Q \\ \lambda & \mapsto & (q(\lambda), p(\lambda)) & \mapsto & q(\lambda), \end{array}$$

$$W : Q \times \mathbb{R}^k \to \mathbb{R} \qquad (q, u) \mapsto W(q, u)$$

$$\Lambda = \left\{ p = \frac{\partial W}{\partial q}(q, u), \quad 0 = \frac{\partial W}{\partial u}(q, u) \right\}$$

• Now, in the new environment $T^*Q_1 \times T^*Q_2$, a version of Maslov-Hörmander's th. goes in this line:

$$\Lambda = \mathrm{graph}(f) \cong T^*Q_1 \xhookrightarrow{j} T^*Q_1 \times T^*Q_2 \cong T^*(Q_1 \times Q_2) \xrightarrow{\pi_{Q_1 \times Q_2}} Q_1 \times Q_2$$

$$\lambda = (q_{(1)}, p_{(1)}) \mapsto (q_{(1)}, p_{(1)}; f_q(\lambda), f_p(\lambda)) \cong (q_{(1)}, f_q(\lambda); p_{(1)}, f_p(\lambda)) \mapsto (q_{(1)}, f_q(\lambda))$$

$$W : Q_1 \times Q_2 \times \mathbb{R}^k \to \mathbb{R} \qquad (q_1, q_2, u) \mapsto W(q_1, q_2, u)$$

$$\mathrm{graph}(f) = \left\{ p_1 = -\frac{\partial W}{\partial q_1}(q_1, q_2, u), \quad p_2 = \frac{\partial W}{\partial q_2}(q_1, q_2, u), \quad 0 = \frac{\partial W}{\partial u}(q_1, q_2, u) \right\}$$

• **(i) Canonical Transformations send Lagrangian submanifolds into Lagrangian submanifolds:**

Theorem *Let*

$$f : (M, \omega) \longrightarrow (N, \bar{\omega})$$

be a symplectomorphism, $f^*\bar{\omega} = \omega$*,*
and

$$j : \Lambda \hookrightarrow (M, \omega)$$

an embedded Lagrangian submanifold. Then

$$f \circ j(\Lambda)$$

is Lagrangian in $(N, \bar{\omega})$*.*

Proof.

$$\bar{\omega}\big|_{f \circ j(\Lambda)} = (f \circ j)^* \bar{\omega} = j^* \circ \underbrace{f^* \bar{\omega}}_{= \omega} = j^* \omega = \omega\big|_{\Lambda} = 0$$

□

(ii) The Composition Rule

$$\begin{array}{ccccccc} T^*M & \overset{CT_1}{\longrightarrow} & T^*M & , & T^*M & \overset{CT_2}{\longrightarrow} & T^*M \\ (x_0, p_0) & \mapsto & (\bar{x}_1, \bar{p}_1) & & (x_1, p_1) & \mapsto & (x_2, p_2), \end{array}$$

Given two Generating Functions:

$$W_1(x_0, \bar{x}_1; u) \quad \text{for} \quad CT_1 : T^*M \to T^*M$$

$$W_2(x_1, x_2; v) \quad \text{for} \quad CT_2 : T^*M \to T^*M$$

then the canonical transformation $CT_{21} = CT_2 \circ CT_1$ is generated by

$$W_{21}(x_0, x_2; u, v, w) := W_1(x_0, w; u) + W_2(w, x_2; v)$$

The Composition Rule

$$\begin{array}{ccccccc} T^*M & \overset{CT_1}{\longrightarrow} & T^*M & , & T^*M & \overset{CT_2}{\longrightarrow} & T^*M \\ (x_0, p_0) & \mapsto & (\bar{x}_1, \bar{p}_1) & & (x_1, p_1) & \mapsto & (x_2, p_2), \end{array}$$

Proof. $\quad W_{21}(x_0, x_2; u, v, w) := W_1(x_0, w; u) + W_2(w, x_2; v)$

$$p_0 = -\frac{\partial W_{21}}{\partial x_0} \quad p_2 = \frac{\partial W_{21}}{\partial x_2} : \quad p_0 = -\frac{\partial W_1}{\partial x_0}(x_0, w; u) \quad p_2 = \frac{\partial W_2}{\partial x_2}(w, x_2; v)$$

$$0 = \frac{\partial W_{21}}{\partial u} \qquad 0 = \frac{\partial W_{21}}{\partial v} : \qquad 0 = \frac{\partial W_1}{\partial u} \qquad 0 = \frac{\partial W_2}{\partial v}$$

$$0 = \frac{\partial W_{21}}{\partial w} : \qquad 0 = \frac{\partial W_1}{\partial \bar{x}_1} + \frac{\partial W_2}{\partial x_1}$$

that is: $\quad 0 = \underbrace{\bar{p}_1}_{\text{the 'final' impulse of } TC_1} - \underbrace{p_1}_{\text{the 'starting' impulse of } TC_2} \qquad \square$

A little algebra for Canonical Transformations - 4

- **The Identity**
The generating function for the trivial canonical transformation 'identity' is given by

$$W_{\mathrm{Id}}(x, X; u) := (X - x) \cdot u$$

- **The Inverse**
Given a Generating Function $W(x, X; u)$ for CT, then $(CT)^{-1}$ is generated by

$$W^{(-1)}(X, x; u) := -W(x, X; u)$$

• **The Characteristics Methods**
Let

$$H : T^*Q \to \mathbb{R}$$

and a real number E s.t. $H^{-1}(E) \neq \emptyset$, better: $\mathrm{rk}\, dH\big|_{H^{-1}(E)} = 1$,
• a **classical** (C^1) **solution** $S(q)$ of the related H-J equation

$$H(q, \frac{\partial S}{\partial q}(q)) = E$$

• (if there exists...maybe just only local... and so on)
• can be thought as an
exact Lagrangian submanifold $\Lambda = im(dS)$ **globally transverse** to the fibers of $\pi_Q : T^* \to Q$

$$\Lambda = im(dS) \subset H^{-1}(E)$$

- How to (geometrically) generalize ?
- **Def. - Geometrical solutions of H-J:** We say that a Λ, Lagrangian in T^*Q, is a geometrical solution of H-J $H = E$ if

$$\Lambda \subset H^{-1}(E)$$

- Recalling dimensions:

$$\dim Q = n, \qquad \underbrace{\Lambda}_{n} \subset \underbrace{H^{-1}(E)}_{2n-1} \subset \underbrace{T^*Q}_{2n}$$

- By relaxing transversality, we accept the possible 'multivalued' character of the Lagrangian submanifolds as solutions of H-J

- We are saying **nothing** now about the topology of j: immersion/embedding,..., $j(\Lambda)$ could be dense into...

Geometrical synopsis of Hamilton-Jacobi equation

- What's the **recipe** to build Lagrangian submanifolds Λ into $H^{-1}(E)$?

- The 2-form ω is represented by the skw $2n \times 2n$ matrix

$$\mathbb{E} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}, \quad \mathbb{E}^T = \mathbb{E}^{-1} = -\mathbb{E} \quad \text{and} \quad \mathbb{E}^2 = -\mathbb{I},$$

- consider the **Hamiltonian vector field** X_H related to H:
- it is defined as an **equality between 1-forms**:

$$i_{X_H}\omega = -dH$$

$$\left\langle \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix} \begin{pmatrix} X_H^q \\ X_H^p \end{pmatrix}, \cdot \right\rangle = -\left\langle \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \cdot \right\rangle$$

$$\Rightarrow \qquad X_H = \begin{pmatrix} X_H^q \\ X_H^p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

- **Theorem (Origin of Characteristics Method)** If the Lagrangian $j : \Lambda \hookrightarrow T^*Q$ solves geometrically H-J: $H = E$, that is $\Lambda \subset H^{-1}(E)$, then the **Hamiltonian vector field is tangent** to Λ:

$$X_H(j(\lambda)) \in T_\lambda\Lambda \qquad \forall \lambda \in \Lambda$$

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$$X_H(j(\lambda)) \in T_\lambda\Lambda \qquad \forall\lambda \in \Lambda$$

- Proof. Since any tangent vector to Λ is also (we adopt trivial identifications: $Dj(\lambda)v \approx v$)) a tangent vector to $H^{-1}(E)$, it is on the kernel of dH,

$$v \in T_\lambda\Lambda \qquad \Rightarrow \quad dH(j(\lambda))v = 0$$

$$i_{X_H}\omega = -dH \qquad i_{X_H}\omega\, v = -dH\, v = 0$$

$$\omega(X_H, v) = 0 \quad \forall v \in T_\lambda\Lambda$$

$\Rightarrow \quad X_H$ is ω-orthogonal to $T_\lambda\Lambda$; since
(i) the space of the vectors ω-orthogonal to $T_\lambda\Lambda$ is of dimension $2n - n = n$ (ω is not degenerate), and since
(ii) all the vectors of $T_\lambda\Lambda$ are ω-orthogonal to $T_\lambda\Lambda$ itself ($j^*\omega = 0$), necessarily X_H is in $T_\lambda\Lambda$.

Characteristics Method for Hamilton-Jacobi equation

- As a consequence:
- take into $H^{-1}(E)$ a n-1-submanifold ℓ_0,

$$\underbrace{\ell_0}_{n-1} \subset \underbrace{H^{-1}(E)}_{2n-1}$$

- such that: $X_H \notin T\ell_0$ (Transversality Condition)
- the candidate solution 'starting' from ℓ_0 is

$$\Lambda = \bigcup_{\lambda \in \mathbb{R}} \phi^{\lambda}_{X_H}(\ell_0)$$

- $\Rightarrow$ dimension is correct (i.e. n),
- $\Rightarrow$ and surely $\Lambda \subset H^{-1}(E)$, from the conservation of H along $\phi^{\lambda}_{X_H}$
- $\Rightarrow$ at the end, we can also check that it is effectively Lagrangian: $\omega|_{\Lambda} = 0$

- **The evolutive case** Let Q be a smooth, connected and closed (i.e: compact & $\partial Q = \emptyset$) manifold.
- Take a Hamiltonian

$$H : \mathbb{R} \times T^*Q \to \mathbb{R}$$

- The Classical Cauchy Problem $\qquad (H \in C^2,\ \sigma \in C^1)$:

$$(Cauchy\ Pr.) \begin{cases} \frac{\partial S}{\partial t}(t,q) + H\left(t, q, \frac{\partial S}{\partial q}(t,q)\right) = 0, \\ \\ S(0,q) = \sigma(q), \end{cases}$$

- We proceed by a space-time 'homogeneization':
-

$$\mathcal{H} : T^*(\mathbb{R} \times Q) \longrightarrow \mathbb{R}$$

$$(t, q; \tau, p) \longmapsto \mathcal{H} := \tau + H(t, q, p)$$

- with the symplectic form on $T^*(\mathbb{R} \times Q)$:

$$\omega = d\tau \wedge dt + dp \wedge dq$$

Characteristics Method for **evolutive** Hamilton-Jacobi equation

- the evolutive H-J $\frac{\partial S}{\partial t}(t,q) + H(t,q,\frac{\partial S}{\partial q}(t,q)) = 0$ reads:
-
$$\mathcal{H} = 0$$
- take into $\mathcal{H}^{-1}(0)$ the following n-submanifold ℓ_0,
$$\underbrace{\ell_0}_{n} \subset \underbrace{\mathcal{H}^{-1}(0)}_{2n+1} \subset \underbrace{T^*(\mathbb{R}\times Q)}_{2n+2}$$
- ℓ_0 **encodes the initial data**:
$$\ell_0 := \left\{ \left(0, q, -H\left(0, q, \frac{\partial \sigma}{\partial q}(q)\right), \frac{\partial \sigma}{\partial q}(q)\right) : \; q \in Q \right\} \subset \mathcal{H}^{-1}(0) \subset T^*(\mathbb{R}\times Q)$$
-
$$\text{the flow } \phi^t_{X_\mathcal{H}} \text{ of } X_\mathcal{H} \text{ is `substantially' the flow of } X_H : \begin{cases} \dot{t} = & 1 \\ \dot{q} = & \frac{\partial H}{\partial p} \\ \dot{\tau} = & -\frac{\partial H}{\partial t} \\ \dot{p} = & -\frac{\partial H}{\partial q} \end{cases}$$

• the $(n+1)$-dimensional **Lagrangian submanifold, geometrical solution** of the Cauchy Problem for $t \in [0, T]$, is

•
$$\Lambda = \bigcup_{t \in [0,T]} \phi^t_{X_{\mathcal{H}}}(\ell_0) \subset T^*(\mathbb{R} \times Q)$$

• Some remarks:

• Λ is the collection of the **wave front sets** at any $t \in [0, T]$:

$$\phi^t_{X_{\mathcal{H}}}(\ell_0)$$

• Furthermore, the $\phi^t_{X_{\mathcal{H}}}(\ell_0)$ are Lagrangian submanifolds $\mathbf{in}$ T^*Q

Generating Functions for **evolutive** Hamilton-Jacobi equation

- Generating Function for Λ?

• Under suitable conditions, from the above
Lagrangian solution Λ, we have to provide a global Generating Function

- if

$$W_t(q_0, q_1; u)$$

is a global Generating Function for the symplectomorphism
$\phi^t_{X_{\mathcal{H}}} : T^*Q \to T^*Q$,

• then:

$$S_t(q;\ \underbrace{u, \xi}_{aux.\ p.}\ ) := \underbrace{\sigma(\xi)}_{\text{g.f. of im}(\mathrm{d}\sigma)} + \underbrace{W_t(\xi, q; u)}_{\text{Geometric Propagator, Green kernel}}$$

• is generating the wave front set for $t \in [0, T]$: $\phi^t_{X_{\mathcal{H}}}(\ell_0)$

- (Overcaming) Drawback: note that for $W_t(\xi, q; u)$, and so for $S_t(q; u, \xi)$, the dimension k of the space of the aux. par., $\mathbb{R}^k$, is depending of t, growing with t.

- A new strategy:

(i) To provide a Generating Function $S_t(q; \overbrace{\xi, u}^{v:=})$ for Λ with a space $v \in \mathbb{R}^k$, k **uniform** (independent) of $t \in [0, T]$

(ii) under suitable hypotheses on H and σ, for any fixed $(t, q) \in [0, T] \times Q$, to pick out **a well precise (among many) critical value** for S,

-

$$\frac{\partial S_t}{\partial v}(q; v) = 0$$

- call it:

$$\mathcal{S}(t, q)$$

- this will be the candidate weak function we are looking for!

Variational solutions of Hamilton-Jacobi equations - 3
Relative Cohomology, Palais-Smale, Min-max solutions

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To sum up:

- Maslov-Hörmander theorem claims that (locally) every Lagrangian submanifold admits Generating Functions $W(q,\xi):\ p = W_{,q},\ \ 0 = W_{,\xi}$
- There exist **three** operations linking (again locally) **all** the Generating Functions for a same Lagrangian submanifold
- Now, our task is to derive, from a Generating Function of a Lagrangian submanifold geom. sol. of H-J equ., a suitable weak (true) function

- we have to pick out, to select, Hamiltonians providing H-J equ. and relative geom. solutions with a
- (i) **unique global** Generating Function,
- and (ii) such that it admits, for any q, a well precise (universal, in a sense) **critical value** W^*: $0 = W_{,\xi}$

Aims and motivation

• (i) **unique global** Generating Function?
• $\rightarrow$ by Amman-Conley-Zehnder method (a sort of Lyapunov-Schmidt with a Fourier cut-off) or
• $\rightarrow$ by Chaperon method (said of broken geodesics) surely Hamiltonians with quadratic p-dependence and $q \in M$ compact with $\partial M = \emptyset$, admit unique global Generating Function
• more, these last Generating Functions are **Quadratic at Infinity (GFQI)**:

$$\text{for} \quad |\xi| > C \text{ (large)} : \quad W(q,\xi) = \xi^t A \xi, \qquad \det A \neq 0$$

• they are Palais-Smale,
• so min-max and Lusternik-Schnirelman theory does work:

• and finally a well precise –the above point (ii)– **min-max critical value** can be achieved.

• **DEF 1** W is GFQI iff:

$$\text{for } \ |\xi| > C \text{ (large)} : \quad W(q,\xi) = \xi^t A\xi, \qquad \det A \neq 0$$

• A generalization of the above def., introduced by Viterbo and studied in detail by Theret, is the following:
• **DEF 2** A generating function $W : M \times \mathbb{R}^k \to \mathbb{R}$, $(q,\xi) \mapsto S(q,\xi)$, is asymptotically quadratic if for every fixed $q \in M$

$$||W(q,\cdot) - \mathcal{P}^{(2)}(q,\cdot)||_{C^1} < +\infty, \tag{1}$$

where $\mathcal{P}^{(2)}(q,\xi) = \xi^t A(q)\,\xi + b(q)\cdot\xi + a(q)$ and $A(q)$ is a nondegenerate quadratic form.

• The two defs are **equivalent**, up to the above three operations!

Synopsis on min-max and Lusternik-Schnirelman theory by Relative Cohomology

•
Let f be a C^2 function on a manifold X. We shall assume either that X is compact or that f satisfies the **Palais-Smale** condition:

• **P-S** *Any sequence* (x_n) *such that* $f'(x_n) \to 0$ *and* $f(x_n)$ *is bounded has converging subsequence.*

• Note that if x is the limit of such a subsequence, it is a critical point of f.
• The aim of Lusternik-Schnirelman theory (L-S theory. , for short) will be to give a lower bound to the set of critical points of f on X in terms of the **topological complexity** of X.
• We denote the sub-level sets by $X^a = \{x \in X | f(x) \leq a\}$.

• We now define this **topological complexity** in terms of **cohomology**

• The idea of utilizing forms in order to construct critical values of f comes back to Birkhoff and Morse.

• Let $Y \subset X$ be two manifolds, $\iota : Y \hookrightarrow X$. Define the complex of **relative** forms

$$\Omega^q(X, Y) = \Omega^q(X) \oplus \Omega^{q-1}(Y)$$

and the following **relative** exterior differential (we will keep using the symbol d to indicate it)

$$d^q : \Omega^q(X) \oplus \Omega^{q-1}(Y) \longrightarrow \Omega^{q+1}(X) \oplus \Omega^q(Y)$$

$$d(\omega, \theta) \colon = (d\omega, \iota^*\omega - d\theta) \in \Omega^{q+1}(X) \oplus \Omega^q(Y).$$

• The relative form (ω, θ) is **relatively closed** if

$$d(\omega, \theta) = (d\omega, \iota^* \omega - d\theta) = (0, 0)$$

that is, if ω is closed in X,
its restriction to Y is exact, and θ is a primitive.

• The relative form (ω, θ) is **relatively exact** if there exists $(\bar{\omega}, \bar{\theta}) \in \Omega^{q-1}(X) \oplus \Omega^{q-2}(Y)$ such that $d(\bar{\omega}, \bar{\theta}) = (\omega, \theta)$, more precisely, $\omega = d\bar{\omega}$ and $\theta = \iota^* \bar{\omega} - d\bar{\theta}$.

• Observe that $d^2 = 0$:

$$d^2(\omega, \theta) = d(d\omega, \iota^* \omega - d\theta) = (d^2\omega, \iota^* d\omega - d(\iota^* \omega - d\theta)) = (0, 0).$$

- The relative cohomology is by definition the space of quotients

$$H^q(X,Y) = \frac{\mathrm{Ker}\ d^q}{\mathrm{Im}\ d^{q-1}} = \frac{Z^q(X,Y)}{B^q(X,Y)}.$$

Using the notation

$$B^*(X,Y) = \bigoplus_{q\geq 0} B^q(X,Y), \qquad H^*(X,Y) = \bigoplus_{q\geq 0} H^q(X,Y), \quad \text{etc.}$$

The elements of $H^*(X,Y)$ are equivalence classes of elements $(\omega,\theta) + B^*(X,Y)$, with $(\omega,\theta) \in Z^*(X,Y)$.

We have seen that ω must be closed in X and exact in Y with θ a primitive.

• **Theorem 1** Let X, X', Y, Y' be manifolds, $f : Y \to X$ an application (e.g. an embedding $f : Y \hookrightarrow X$) and

$$\varphi : X \to X', \qquad \psi : Y \to Y'$$

two diffeomorphisms. Define $f' := \varphi \circ f \circ \psi^{-1}$,

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\varphi"] \\ Y' \arrow[r, "f'"'] & X' \end{tikzcd}$$

Then

$$H^*(X, Y) \equiv H^*(X', Y')$$

(invariance by diffeomorphisms)

• **Theorem 2** For every diffeomorphism $f : Y \to X$, one has $H^*(X, Y) = 0$.

Proof.
One can apply theorem 1,

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "f"'] & X \arrow[d, "\mathrm{id}_X"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

and observe that the closed forms on X, that also are exact on X, vanish in $H^*(X, X) = 0$. □

(trivial cohomology between diffeomorphic manifolds)

- **Theorem 3** *Let* $Z \subseteq Y \subseteq X$*,* i *and* j *be the inclusions:*

$$Z \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} X$$

The sequence

$$H^*(X,Y) \xrightarrow{\underline{i}} H^*(X,Z) \xrightarrow{\underline{j}} H^*(Y,Z)$$

is exact, which means: $\mathrm{Im}\ \underline{i} = \mathrm{Ker}\ \underline{j}$.

- *Proof.* The map $\underline{i}$ takes a (ω, θ) in $H^*(X,Y)$ and maps it to an element of $H^*(X,Z)$ by restricting the domain of θ, from Y to Z.

The map $\underline{j}$ takes an (ω, θ) in $H^*(X,Z)$ and maps it in $H^*(Y,Z)$ by restricting the domain of ω, from X to Y.

The kernel of $\underline{j}$ are all the closed forms ω in X, that vanish (so that are exact, think of equivalence classes) in Y, and hence in Z.

The image of $\underline{i}$ are all the closed forms ω in X, that are exact in Y, hence remaining exact after restriction to Z. $\square$

- In summary: Relative Cohomology is invariant
- (i) under **diffeomorphisms**,
- and also
- (ii) under **retractions**:

Given $\iota : S \hookrightarrow X$, S is a retract of X if $\exists$ a continuous map (called retraction) $r : X \to S$ such that $r(y) = y,\ \forall y \in S$.
In other terms: $r \circ \iota = \mathrm{id}_S$, that is the inclusion ι admits a continuous left inverse,

$$S \overset{\iota}{\hookrightarrow} X \overset{r}{\longrightarrow} S$$

$$r \circ \iota = \mathrm{id}_S$$

- (iii) under **excisions**:

$\exists$ isomorphism

$$j^* : H^*(X, Y) \longrightarrow H^*(X \setminus U, Y \setminus U)$$

if the open U is disjoint from the boundary of Y, then U can be eliminated without changing cohomology

Synopsis on min-max and Lusternik-Schnirelman theory by Relative Cohomology

- Take the pair (f, X) P-S, $\quad f : X \to \mathbb{R}$
- Take $a < b$,
- Consider X^b, $\quad X^a$, and $f^{-1}[a,b] = \overline{X^b \setminus X^a}$
- **Suppose no critical value** of f in $[a,b]$
- $\Rightarrow$ **Theorem:** *X^b and X^a are diffeomorphic.*
- $\Rightarrow$ by (invariance by diffeomorphisms)

$$H^*(X^b, X^a) = 0$$

- A sketch of proof of the above Theor:

$\nabla f \neq 0$ in $f^{-1}[a,b]$, by the flow of a vector field, which in $f^{-1}[a,b]$ is $\mathcal{X} = -\frac{\nabla f}{\|\nabla f\|^2}$, $\quad \frac{d}{dt} f \circ \phi^t_{\mathcal{X}} = \nabla f \cdot \mathcal{X} = -1$, so

$$\phi^{b-a}_{\mathcal{X}} : X^b \to X^a$$

is the diffeomorphism we are looking for. □

- (Rem: by P-S, $\|\nabla f\|$ is 'bounded away from zero', so $\mathcal{X}$ is Lip)

Synopsis on min-max and Lusternik-Schnirelman theory by Relative Cohomology

- Thus, something *more interesting* may occur in $X^b \setminus X^a$ if there exists some **non vanishing** class $\alpha \neq 0$ in $H^*(X^b, X^a)$
- More precisely, the following Theorem holds:

- **Theorem (min-max)** *Let $\alpha \neq 0$ in $H^*(X^b, X^a)$. For any $a \leq \lambda \leq b$ we write:*

$$\iota_\lambda : X^\lambda \hookrightarrow X^b$$

and denote the induced map between relative cohomologies by

$$\iota_\lambda^* : H^*(X^b, X^a) \to H^*(X^\lambda, X^a)$$

- *(Note that: $\iota_b^* \alpha = \alpha$, and $\iota_a^* \alpha = 0$)*
- Then

$$c(\alpha, f) := \inf \left\{ \lambda \in [a, b] : \ \iota_\lambda^* \alpha \neq 0 \right\}$$

- *is a* **critical value** *for f.*

•

Proof.
By contradiction: for $\alpha \in H^*(X^b, X^a), \alpha \neq 0$, the value $c(\alpha, f)$ is a **regular value** for f.
• $(PS) \Rightarrow$ the set of critical points of f in $f^{-1}([a,b])$ is a compact set, then closed. There exists a ε (small) such that $[c-\varepsilon, c+\varepsilon]$ does not contain critical values[4] of f. Hence, in view of an above theorem,

$$H^*(X^{c+\varepsilon}, X^{c-\varepsilon}) = 0$$

• Consider now the **exact sequence** based on: $X^a \subseteq X^{c-\varepsilon} \subseteq X^{c+\varepsilon}$ (rem. Th. 3 above):

$$\begin{tikzcd}
0 = H^*(X^{c+\varepsilon}, X^{c-\varepsilon}) \arrow[r] & H^*(X^{c+\varepsilon}, X^a) \arrow[r, "\star"] & H^*(X^{c-\varepsilon}, X^a) \\
& \alpha \in H^*(X^b, X^a) \arrow[u, "i^*_{c+\varepsilon}"'] &
\end{tikzcd}$$

• Since the horizontal sequence is exact, one has that the kernel of $\star$ is the null space, hence $\star$ is injective. By definition of c, $\alpha \neq 0$ in $H^*(X^{c+\varepsilon}, X^a)$, hence its image under the map $\star$ should be non-zero: $\alpha \neq 0$ in $H^*(X^{c-\varepsilon}, X^a)$, this fact contradicts the definition of c. □

[4] in other words, c cannot be an accumulation point of critical values

It is time to come back to GFQI

-

• We will see that,
at least for Hamiltonians which are quadratic (a generic hyperbolic q.f.) on p's and with possible compactly supported 'perturbation' on $[0,T]\times T^*\mathbb{T}^n$, like:

$$H=\frac{1}{2}p^tAp+V(t,q,p)$$

• the Lagrangian submanifold Λ, geometrical solution of the Cauchy Problem for **The evolutive case**

$$(Cauchy\ Pr.)\begin{cases}\frac{\partial S}{\partial t}(t,q)+H\left(t,q,\frac{\partial S}{\partial q}(t,q)\right)=0,\\ S(0,q)=\sigma(q),\end{cases}$$

•

is generated by a Generating Function Quadratic at Infinity,

$$S^t(q;\xi,U),\ \text{with respect to the aux. parameters }\ (\xi,U)$$

Quadraticity at infinity: uniqueness & Palais-Smale

- A Theorem by Viterbo globalizes to the GFQI the already known (local) theorem characterizing, by three operations, all the (local) GF of a **same** Lagrangian submanifold Λ.
- $\Rightarrow$ In essence: the GFQI are unique, up to the three operations

- Together with uniqueness, we gain also the following crucial property:

- GFQI are Palais-Smale

- This is a crucial step in order to define the minmax or variational solution of H-J

- **Theorem** *Let $f : M \times \mathbb{R}^k \to \mathbb{R}$, $f : (q, \xi) \mapsto f(q, \xi)$ be a GFQI. Then, for any fixed q, $f(q, \cdot)$ is Palais-Smale.*

- *Proof.* For every fixed q, let $\{\xi_j\}_{j \in \mathbb{N}}$ be a sequence such that

$$|f(q, \xi_j)| \leq \bar{C} < +\infty, \qquad \lim_{j \to +\infty} \frac{\partial f}{\partial \xi}(q, \xi_j) = 0$$

If the sequence $\{\xi_j\}_{j \in \mathbb{N}}$ is, from a certain index on, in a compact set Ω, then there must be a converging subsequence, let say that $\bar{\xi}$ is its limit. This limit must obviously be a critical point. Let us verify that nothing different can happen. Since f is a GFQI, then for $|\xi| > C$, $f(q, \xi) = \xi^T A \xi$, where $\xi^T A \xi$ is a non-degenerate quadratic form. If there were only finite terms of the sequence in some Ω compact set, it would follow that $\lim_{j \to +\infty} |\xi_j| = +\infty$. Then the terms ξ_j would end up outside from the ball $B(C)$, and this would contradicts the hypothesis, since in such case

$$\frac{\partial f}{\partial \xi}(q, \xi_j) = 2A\xi_j$$

Recalling that A is non-degenerate, $\frac{\partial f}{\partial \xi}(q, \xi_j)$ would then tend to ∞ and not to zero.

□

GfQI: sub-level sets for great $|c|$

- Let $f(q,\xi)$ be a GFQI:
  if $|\xi| > K$ then $f(q,\xi) = \xi^t A \xi$ with $A^t = A$ non-degenerate.
  Let R be the *spectral radius* of A, i.e.the supremum of the absolute value $|\lambda|$ of the eigenvalues λ of A, $A\xi_\lambda = \lambda \xi_\lambda$,

$$-R\,|u|^2 \leq \xi^t A \xi \leq R\,|u|^2.$$

If for **chosen (large enough)** $c > 0$ such that

$$-c < \min_{\xi \in B(K)} f(\xi) \leq \max_{\xi \in B(K)} f(\xi) < c, \qquad \text{and} \qquad R\,K^2 < c,$$

then

$$f^c = A^c, \qquad f^{-c} = A^{-c}$$

hence:

$$\boxed{H^*(f^c, f^{-c}) = H^*(A^c, A^{-c})}$$

•

We utilize a result from algebraic geometry:

it is well know that the relative cohomology for A is (see next paragraph for further clarifications)

$$H^h(A^c, A^{-c}) = \begin{cases} \mathbb{R}, & \text{if } h = i, \ \textit{Morse index } (: \# \text{ of neg. eigenvalues) of } A, \\ 0, & \text{if } \ h \neq i. \end{cases}$$

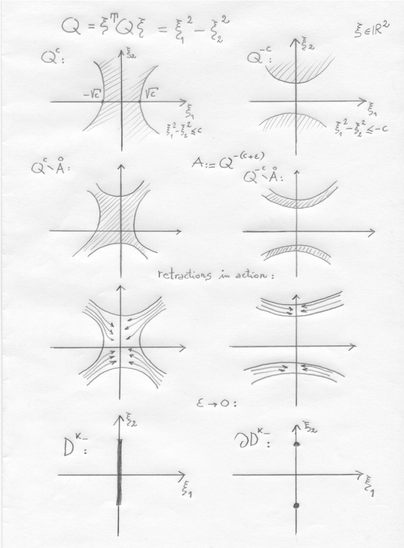
Let α be *precisely* the generator of the 1-dimensional $H^i(A^c, A^{-c})$.
We define the Variational min-max solutions for H-J equations:

$$\mathcal{S}(t,q) := c(\alpha; S(t,q;\cdot))$$

• Proceeding in this way for every (t,q), the solution defined with this technique is known as the *min-max*, or *variational* solution, by Chaperon Sikorav Viterbo.
It comes out that it is a Lipschitz-continuous function (see the unpublished work of Ottolenghi-Viterbo) and the beautiful book of Siburg. This last fact is rather surprising, it is the same regularity of the viscosity solutions.

Interlude: Relative Cohomology of quadratic forms

- Rem: $Q := \xi^T Q \xi, \qquad Q^c := \{\xi \in \mathbb{R}^N : Q \leq c\}, \qquad A := Q^{-(c+\varepsilon)}$
- A 'graphical' explanation of $H^*(Q^c, Q^{-c}) \cong H^*(D^{k_-}, \partial D^{k_-})$:



- $A := Q^{-(c+\varepsilon)}$,

We have seen:

$$H^*(Q^c, Q^{-c}) \cong_{by\ excision} H^*(Q^c \setminus \mathring{A}, Q^{-c} \setminus \mathring{A}) \cong_{by\ retraction} H^*(D^{k_-}, \partial D^{k_-})$$

- $H^*(D^{k_-}, \partial D^{k_-}) = H_c^*(D^{k_-}, \partial D^{k_-}) \cong H_c^*(D^{k_-} \setminus \partial D^{k_-}, \emptyset) \cong H_c^*(\mathring{D}^{k_-})$,
- $H_c^*(\mathring{D}^{k_-}) \cong H_c^*(\mathbb{R}^{k_-})$,

A classical theorem says: $$H_c^p(\mathbb{R}^{k_-}) = \begin{cases} \mathbb{R}, & \text{if } p = k_- \\ 0, & \text{if } p \neq k_- \end{cases}$$

- finally:

$$H^*(Q^c, Q^{-c}) \cong H^*(D^{k_-}, \partial D^{k_-}) \cong \mathbb{R}$$

□

Variational solutions of Hamilton-Jacobi equations - 4
Amann-Conley-Zehnder reduction - Minmax & viscosity

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• Here the construction of a global generating function for the geometric solution for $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ on $T^*\mathbb{R}^n$
(then, the case $\frac{1}{2}p^T Bp + V$, B hyperbolic) :
•

$$(CP)_H \begin{cases} \frac{\partial S}{\partial t}(t,q) + \frac{1}{2}\left|\frac{\partial S}{\partial q}(t,q)\right|^2 + V(q) = 0 \\ S(0,q) = \sigma(q), \end{cases}$$

• Let us consider the set of curves:

$$\Gamma := \left\{\gamma(\cdot) = (q(\cdot), p(\cdot)) \in H^1\left([0,T], \mathbb{R}^{2n}\right) : \ p(0) = d\sigma(q(0))\right\}$$

• Sobolev imbedding theorem, $H^1\left((0,T), \mathbb{R}^{2n}\right) \hookrightarrow C^0\left([0,T], \mathbb{R}^{2n}\right)$
• The candidate gen. funct. is the **Hamilton-Helmholtz functional Action**:

$$A : [0,T] \times \Gamma \longrightarrow \mathbb{R}$$

$$(t, \gamma(\cdot)) \mapsto A[t, \gamma(\cdot)] := \sigma(q(0)) + \int_0^t [p(r) \cdot \dot{q}(r) - H(r, q(r), p(r))]\, dr$$

A ∞-parameter Generating Function

- $$A : [0,T] \times \Gamma \longrightarrow \mathbb{R}$$
- Since $\dot{\gamma} = \Phi \quad (\text{velocities}) \in L^2$,
- we introduce the following **bijection** representation g for $[0,T] \times \Gamma$:

$$\begin{aligned} g : [0,T] \times \mathbb{R}^n \times L^2\left((0,T), \mathbb{R}^{2n}\right) &\longrightarrow [0,T] \times \Gamma \\ (t, q, \Phi) \longmapsto g(t, q, \Phi) &= (t, \gamma(\cdot)), \qquad \gamma(\cdot) = \gamma_{t,q}(\cdot) \end{aligned}$$

$$\Phi = (\Phi_q, \Phi_p)$$

- $$\gamma(s) := \left(q - \int_s^t \Phi_q(r)\,dr,\ \ \frac{\partial \sigma}{\partial q} \underbrace{\left(q - \int_0^t \Phi_q(r)\,dr \right)}_{q(0)} + \int_0^s \Phi_p(r)\,dr \right)$$
-

To be more clear, we remark that the second value of the map $g(t, q, \Phi)$ is the curve $\gamma(\cdot) = (q(\cdot), p(\cdot))$ which is
1) starting from $(q(0), d\sigma(q(0)))$, such that
2) $\dot{\gamma}(\cdot) = \Phi(\cdot)$, and
3) $q(t) = q$.

A ∞-parameter Generating Function

- The geometrical solution is realized by the web of the characteristics coming out from the n-dim *initial manifold*:

$$(\Gamma_H)_\sigma = \left(0, q; -H\left(0, q, \frac{\partial \sigma}{\partial q}(q)\right), \frac{\partial \sigma}{\partial q}(q)\right) \subset T^*\mathbb{R}^{n+1}$$

- **Theorem** *The infinite-parameter generating function:*

$$W = A \circ g : [0,T] \times \mathbb{R}^n \times L^2 \longrightarrow \mathbb{R}, \tag{2}$$

$$(t, q, \Phi) \longmapsto W(t, q, \Phi) := A \circ g(t, q, \Phi),$$

generates $L_H = \bigcup_{0 \le t \le T} \varphi_H^t\left((\Gamma_H)_\sigma\right)$*, the geometric solution for the Hamiltonian* $H(q,p) = \frac{1}{2}|p|^2 + V(q)$*:*

H-J : $$\frac{\partial W}{\partial t}(t,q,\Phi)\Big|_{\Phi:\frac{DW}{D\Phi}(t,q,\Phi)=0} + H\left(t, q, \frac{\partial W}{\partial q}(t,q,\Phi)\Big|_{\Phi:\frac{DW}{D\Phi}(t,q,\Phi)=0}\right) = 0$$

Initial data : $$\frac{\partial W}{\partial q}(0,q,\Phi)\Big|_{\Phi:\frac{DW}{D\Phi}(t,q,\Phi)=0} = \frac{\partial \sigma}{\partial q}(q)$$

- Note: L^2 is the ∞-dimensional space of auxiliary parameters

The finite reduction (A-C-Z method)

- Since $\frac{DW}{D\Phi}(t,q,\Phi)=0$ selects in Γ characteristic curves, we will reduce the L^2-set of $\{(\Phi_q,\Phi_p)\}$ to the smaller L^2-set of the alone $\{\Phi_q\}$: it is substantially the **Legendre transformation** at work.
- $$\frac{DW}{D\Phi}(t,q,\Phi)=0 \approx \begin{cases}\dot q = p\\ \dot p = -\frac{\partial V}{\partial q}(q)\end{cases} \quad \Rightarrow \begin{cases}\Phi_q(s)=\frac{\partial\sigma}{\partial q}\left(q-\int_0^t \Phi_q(r)\,dr\right)+\int_0^s \Phi_p(r)\,dr\\ \underbrace{\Phi_p(s)=-\frac{\partial V}{\partial q}\left(q-\int_s^t \Phi_q(r)\,dr\right)}_{\Phi_p\quad is\ determined\ by\ \Phi_q}\end{cases}$$

Hence

$$\Phi_q(s)=\frac{\partial\sigma}{\partial q}\left(q-\int_0^t \Phi_q(r)\,dr\right)-\int_0^s \frac{\partial V}{\partial q}\left(q-\int_r^t \Phi_q(\tau)\,d\tau\right)dr \qquad (\bullet)$$

- $\Rightarrow$ Here $(\bullet)$ is a **fixed point** problem for $\Phi_q(\cdot)$

• By simplicity, in the following, we set the initial data:

$$\sigma \equiv 0$$

• Actually, this is not restrictive:
Consider the canonical transformation:

$$\begin{cases} q = \tilde{q} \\ p = \tilde{p} + \frac{\partial \sigma}{\partial q}(\tilde{q}) \end{cases} \qquad (\text{easily we see}: \quad dp \wedge dq = d\tilde{p} \wedge d\tilde{q})$$

$$K(\tilde{q}, \tilde{p}) = H(q,p)\Big|_{q=\tilde{q},\ p=\tilde{p}+\frac{\partial \sigma}{\partial q}(\tilde{q})} = H(\tilde{q}, \tilde{p} + \frac{\partial \sigma}{\partial q}(\tilde{q}))$$

If $(\tilde{q}(t), \tilde{p}(t)$ is a characteristic for K, starting from $\tilde{p}(0) = 0$, then

$$(q(t), p(t)) = (\tilde{q}(t), \tilde{p}(t) + \frac{\partial \sigma}{\partial q}(\tilde{q}(t))$$

is a characteristic for H, starting from $p(0) = \frac{\partial \sigma}{\partial q}(q(0))$.

The finite reduction (A-C-Z method)

- For every $\Phi_q \in L^2\left((0,T), \mathbb{R}^n\right)$, Fourier expansion:

$$\Phi_q(s) = \sum_{k\in\mathbb{Z}} \left(\Phi_q\right)_k e^{i(2\pi k/T)s}$$

- For an arbitrarily fixed cut-off $N \in \mathbb{N}$, the projection maps $\mathbb{P}_N$ and $\mathbb{Q}_N$ on the basis $\left\{e^{i(2\pi k/T)s}\right\}_{k\in\mathbb{Z}}$ of $L^2\left((0,T), \mathbb{R}^n\right)$,

$$\mathbb{P}_N\Phi_q(s) := \sum_{|k|\leq N} \left(\Phi_q\right)_k e^{i(2\pi k/T)s}, \qquad \mathbb{Q}_N\Phi_q(s) := \sum_{|k|>N} \left(\Phi_q\right)_k e^{i(2\pi k/T)s}$$

- $$\mathbb{P}_N L^2 \oplus \mathbb{Q}_N L^2 = L^2\left((0,T), \mathbb{R}^n\right)$$

We will write $u := \mathbb{P}_N\Phi_q$ and $v := \mathbb{Q}_N\Phi_q$

$$\Rightarrow \qquad \Phi_q = u + v$$

- **Theorem (Lip-contractive map)**
Let $\sup_{q\in\mathbb{R}^n}\left|V^{''}(q)\right| = C \ (< +\infty)$. **Fix** *the cut-off* N.
For **fixed** $(t,q)\in[0,T]\times\mathbb{R}^n$ *and* **fixed** $u\in\mathbb{P}_N L^2\left((0,T),\mathbb{R}^n\right)$,
the map (try to recall $(\bullet)$*...)*

$$F:\mathbb{Q}_N L^2\left((0,T),\mathbb{R}^n\right)\longrightarrow\mathbb{Q}_N L^2\left((0,T),\mathbb{R}^n\right)$$

$$v\longmapsto\mathbb{Q}_N\left\{-\int_0^s\frac{\partial V}{\partial q}\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}$$

is Lipschitz with constant

$$\mathit{Lip}(F)\leq\frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right)$$

- We will choose N such that $\frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right)<1$
- Denote by $\mathcal{F}(t,q,u)(s)$, shortly $\mathcal{F}(u)$, the **fixed point map**:

$$\mathcal{F}(u)=\mathbb{Q}_N\left\{-\int_0^s\frac{\partial V}{\partial q}\left(q-\int_r^t\left(u+\mathcal{F}(u)\right)(\tau)\,d\tau\right)dr\right\}.$$

The finite reduction (A-C-Z method)

- Recall the above fixed point equation for $\sigma \equiv 0$:

$$\Phi_q(s) = -\int_0^s \frac{\partial V}{\partial q}\left(q - \int_r^t \Phi_q(\tau)\, d\tau\right) dr \qquad (\bullet)$$

-

$\to$ for any u :

$$\mathcal{F}(u) = \mathbb{Q}_N \left\{ -\int_0^s \frac{\partial V}{\partial q}\left(q - \int_r^t (u + \mathcal{F}(u))(\tau)\, d\tau\right) dr \right\} \qquad (*)$$

-

$\to$ search for some u :

$$u = \mathbb{P}_N \left\{ -\int_0^s \frac{\partial V}{\partial q}\left(q - \int_r^t (u + \mathcal{F}(u))(\tau)\, d\tau\right) dr \right\} \qquad (**)$$

• summing m. by m., we restore –and solve– $(\bullet)$:
$q_{t,q}(s) = q - \int_s^t \Phi_q(r)\, dr|_{\Phi_q = u + \mathcal{F}(u)}$
• equation $(**)$ is a **finite dimensional equation**, sometimes said 'bifurcation equation' in some analogous Lyapunov-Schmidt procedure.

- $\mathbb{P}_N L^2((0,T),\mathbb{R}^n) \approx \mathbb{R}^{n(2N+1)}$:

is the (new, finitely reduced) finite-dim. space of aux. parameters u

-

Theorem *The finite-parameter function:*

$$\overline{W} := [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n(2N+1)} \longrightarrow \mathbb{R},$$

$$(t,q,u) \longmapsto \overline{W}(t,q,u) =$$

$$= \left\{ \int_0^t \left[p(s)\cdot \dot{q}(s) - H(s,q(s),p(s)) \right] ds \right\} \Big|_{(q(s),p(s))},$$

where $(q(s),p(s))$ *is obtained by the finite reduction, depending on* t,q,u*:*
$(q(s),p(s)) =$

$$= pr_\Gamma \circ g \left(t, q, \underbrace{[u+\mathcal{F}(u)](s)}_{\Phi_q(s)}, \underbrace{-\frac{\partial V}{\partial q}\left(q - \int_r^t (u+\mathcal{F}(u))(\tau)\, d\tau \right)}_{\Phi_p(s)} \right),$$

generates the geometric solution for $H(q,p) = \frac{1}{2}|p|^2 + V(q)$*.*

The finite reduction (A-C-Z method)

• The last task in order to prove the theorem, is to see that the 'biforcation equation':

$$u = \mathbb{P}_N \left\{ - \int_0^s \frac{\partial V}{\partial q} \left(q - \int_r^t \left(u + \mathcal{F}\left(u\right)\right)(\tau)\, d\tau \right) dr \right\} \qquad (**)$$

is **precisely** given by

$$\frac{\partial \overline{W}}{\partial u}(t, q, u) = 0$$

• The other relations hold:

$$\frac{\partial \overline{W}}{\partial t}(t, q, u) + H(t, q, \frac{\partial \overline{W}}{\partial q}(t, q, u)) = 0 \qquad p(0) = \frac{\partial \overline{W}}{\partial q}(0, q, u) = 0,$$

• Since the fixed point map $\mathcal{F}$ can **also** obtained by the **implict function Th.**, more smoothness is gained for the generating function:

• **Smoothness:** For fixed $(t, q) \in [0, T] \times \mathbb{R}^n$,
$u \mapsto \mathcal{F}(u)$ and $u \mapsto \frac{\partial \mathcal{F}}{\partial u}(u)$ are **uniformly bounded**.

• **Theorem (G.F. Quadratic at ∞):** The finite-parameters function

$$\overline{W} := A \circ g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n(2N+1)} \longrightarrow \mathbb{R},$$

$$(t, q, u) \longmapsto \bar{W}(t, q, u) =$$

$$= \left\{ \int_0^t \left[\frac{1}{2} |\dot{q}(s)|^2 - V(q(s)) \right] ds \right\} \Big|_{q(s) = q - \int_s^t [u(r) + (\mathcal{F}(u))(r)] dr}$$

is asymptotically quadratic: there exists an u-polynomial $\mathcal{P}^{(2)}(t, q, u)$ such that for any fixed $(t, q) \in [0, T] \times \mathbb{R}^n$

$$||\overline{W}(t, q, \cdot) - \mathcal{P}^{(2)}(t, q, \cdot)||_{C^1} < +\infty$$

and, **in this specific mechanical case, its leading term is positive defined (Morse index is 0)**.

The finite reduction (A-C-Z method), the **NON-CONVEX** case

• Whenever the Lagrangian L is non convex, e.g.

$$L(q,\dot q)=\frac{1}{2}\dot q^T B\dot q - V(q), \qquad \text{where B is a generically **hyperbolic** matrix,}$$

• a global Legendre transformation still does work (even thought Young-Fenchel is gone)

• **Theorem (G.F. Quadratic at ∞):** The finite-parameters function

$$\overline{W} := A\circ g : [0,T]\times\mathbb{R}^n\times\mathbb{R}^{n(2N+1)} \longrightarrow \mathbb{R},$$

$$(t,q,u) \longmapsto \bar W(t,q,u) =$$

$$= \left\{\int_0^t \left[\frac{1}{2}\dot q^T B\dot q - V(q(s))\right] ds\right\}\Big|_{q(s)=q-\int_s^t [u(r)+(\mathcal{F}(u))(r)]dr}$$

is asymptotically quadratic: there exists an u-polynomial $\mathcal{P}^{(2)}(t,q,u)$ such that for any fixed $(t,q)\in[0,T]\times\mathbb{R}^n$

$$||\overline{W}(t,q,\cdot)-\mathcal{P}^{(2)}(t,q,\cdot)||_{C^1} < +\infty$$

and **its leading term is has the Morse index $\neq 0$ (it will be related to the Morse index of B)**.

The finite reduction (A-C-Z method)

• Proof (trace, for the **convex case**)
Through the Legendre transformation,

$$\overline{W}\left(t,q,u\right)=\left\{\int_0^t\left[\frac{1}{2}|\dot{q}\left(s\right)|^2-V\left(q\left(s\right)\right)\right]ds\right\}\Big|_{q(s)=q-\int_s^t[u(r)+(\mathcal{F}(u))(r)]dr}$$

$$=\int_0^t\left\{\frac{1}{2}|u\left(s\right)+\left(\mathcal{F}\left(u\right)\right)\left(s\right)|^2-V\left(q-\int_s^t\left[u\left(r\right)+\left(\mathcal{F}\left(u\right)\right)\left(r\right)\right]dr\right)\right\}ds.$$

As a consequence of the compactness of V, of the uniformely boundness of $\mathcal{F}$ and its derivatives, for fixed $(t,q)\in[0,T]\times\mathbb{R}^n$ we obtain that

$$||\overline{W}\left(t,q,\cdot\right)-\mathcal{P}^{(2)}\left(t,q,\cdot\right)||_{C^1}<+\infty,$$

where $\mathcal{P}^{(2)}\left(t,q,u\right)$ is polynomial with **positive defined leading term**

$$\frac{1}{2}\int_0^t|u(s)|^2ds=u^TQu$$

(hence with **Morse index** 0) and linear term with uniformly bounded coefficient, so that, $\overline{W}\left(t,q,u\right)$ is an asymptotically quadratic generating function. $\square$

• Now, given $\overline{W}(t,q,u)$, we construct the variational solution:
• For any fixed (t,q), $u \mapsto \overline{W}(t,q,u)$ is Palais-Smale $\Rightarrow$ L.-S. does work,
• Relative Cohomology of the sub-level sets of $\overline{W}(t,q,u)$ and $u^T Qu$ are equivalent for **large** $c > 0$:

$$H^*(\overline{W}(t,q,\cdot)^c, \overline{W}(t,q,\cdot)^{-c}) \approx H^*(Q^c, Q^{-c})$$

• We recall that the relative cohomology of quadrics is 1-dim:

$$H^h(Q^c, Q^{-c}) = \begin{cases} \mathbb{R}, & \text{if } h = i: \ \textit{Morse index} \ (\# \text{ of neg. eigenvalues}) \text{ of } Q, \\ 0, & \text{if } \ h \neq i. \end{cases}$$

• In the **convex** case we are concerning, we have $i = 0$: $\quad H^0(Q^c, Q^{-c}) = \mathbb{R}$

- Let $\alpha = \mathbf{1}$ be the generator of the 1-dimensional $H^0(Q^c, Q^{-c}) \approx \mathbb{R}$
- (Note that, concerning with the absolute deRham cohomology, for any manifold M with k connected components

$$H^0_{dR}(M) = \mathbb{R}^k$$

This follows from the fact that any smooth function on M with zero derivative (i.e. locally constant) is constant on each of the connected components of M.)
- For large c,

$$H^0(\overline{W}(t,q,\cdot)^c, \overline{W}(t,q,\cdot)^{-c}) = H^0(Q^c, Q^{-c}) = \mathbb{R}$$

but, for suitable small $\lambda < c$, some other connected components can arise for

$$\overline{W}(t,q,\cdot)^\lambda$$

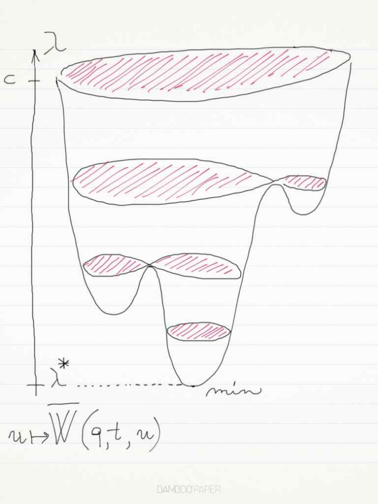
so that

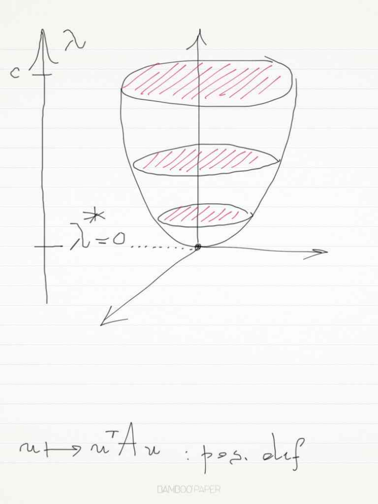
$$H^0(\overline{W}(t,q,\cdot)^\lambda, \overline{W}(t,q,\cdot)^{-\lambda}) \neq H^*(Q^\lambda, Q^{-\lambda})$$

see pictures, $H^0(\overline{W}\,(t,q,\cdot)^{\lambda}\,,\overline{W}\,(t,q,\cdot)^{-\lambda}) = \mathbb{R}^2 = \text{span}(\alpha_1,\alpha_2)$
and, in such a case,

$$\iota_\lambda^* \mathbf{1} = \mathbb{R}(\alpha_1 + \alpha_2)$$

that is, a **same** constant is assigned to both connected components.





- We define the variational min-max solution for H-J:

$$\mathcal{S}(t,q) = \text{minmax}(\overline{W}\,(t,q,;\cdot)) := \inf\{\lambda \in [-c,c] : \quad \iota_\lambda^* \mathbf{1} \neq 0\}$$

- $$\text{Finally : } \Rightarrow \quad \mathcal{S}(t,q) = \min_u \overline{W}\,(t,q,;u)$$

- We have to recall some facts:
- (i) $u \mapsto \overline{W}(t,q,\cdot)$ is a finite reduction of

$$\Phi_q(\cdot) \mapsto \left\{ \sigma(q(0)) + \int_0^t \left[\frac{1}{2} |\dot{q}(s)|^2 - V(q(s)) \right] ds \right\} \Big|_{q(s)=q-\int_s^t \Phi_q(r)dr}$$

- (ii) Critical points of

$$u \mapsto \overline{W} \qquad (finite)$$

are **one-to-one** related to the critical points of

$$\Phi_q \mapsto \sigma + \int L\, ds \qquad (infinite)$$

- (iii) some more is true: Morse indices related to $(infinite)$ are precisely Morse indices related to $(finite)$
- $\Rightarrow$ This is sufficient to say that the variational min-max solution:

$$\mathcal{S}(t,q) = \min_{u \in \mathbb{P}_N L^2} \overline{W}(t,q;u) = \inf_{\tilde{q}(\cdot):\tilde{q}(t)=q} \left\{ \sigma(\tilde{q}(0)) + \int_0^t L(\tilde{q}(s), \dot{\tilde{q}}(s))\, ds \right\}$$

$=$ Lax-Oleinik semi-group : viscosity solution!

-

Lipschitz property

in the following slides: $x = (q, t)$, u : aux. parameters

Theorem (minmax are Lipschitz) *Let $W(x, u)$ be the GFQI for a geometrical solution (a Lagrangian submanifold) for a H-J problem,*

$$W(x, u) = u^T Q u, \qquad |u| > K \ (: \textit{large})$$

Let

$$S(x) = \textit{minmax}\, W(x, \cdot) = \inf\{\lambda \in [-c, c] : \quad \iota_\lambda^* \alpha \neq 0\}$$

where α is the class generator of $H^i(Q^c, Q^{-c})$, i : Morse index of A

be the related variational minmax solution of the H-J equation

$$H(x, \frac{\partial S}{\partial x}(x)) = e$$

Then $S(x)$ is Lipschitz.

- Proof.

Denote by $C > 0$ the Lipschitz constant of the GFQI in $U = \mathbb{T}^n \times [0, T]$, uniformely for $\xi \in \mathbb{R}^k$:

$$C = \sup_{x \in U \ u \in \mathbb{R}^k} \left| \frac{\partial W}{\partial x}(x, u) \right|$$

so that

$$|W(x, u) - W(y, u)| \leq C|x - y| \qquad x, y \in U \tag{$*$}$$

Def.: For fixed x, let now to define, for $\varepsilon > 0$ arbitrary small,

$$a_x(y) := S(x) + \varepsilon + C|x - y|, \quad \forall y \in U$$

- $$a_x(y) := S(x) + \varepsilon + C|x-y|, \quad \forall y \in U$$

$\Big($recall the notation for the sublevel sets: $W_x^c := \{u \in \mathbb{R}^k : W(x,u) \le c\}\Big)$

- We notice that

$$W_x^{a_x(x)} \subseteq W_y^{a_x(y)} \qquad (**)$$

In fact, if $u \in W_x^{a_x(x)}$,

$$W(x,u) \le a_x(x) \underbrace{=}_{\substack{\text{by definition of } a_x(y)\\ \text{for } y=x}} S(x) + \varepsilon$$

from $(*)$,

$$W(y,u) \le W(x,u) + C|x-y| \le S(x) + \varepsilon + C|x-y| \underbrace{=}_{\text{by definition of } a_x(y)} a_x(y).$$

Variational solutions for general Hamiltonians are Lipschitz

• By the very definition of $S(x)$, and $S(x) < S(x) + \varepsilon = a_x(x)$, the relative cohomology $H^*(W_x^{a_x(x)}, W^{-c})$ contains[5] a non vanishing class α, so, by $W_x^{a_x(x)} \subseteq W_y^{a_x(y)}$, the same is true for $H^*(W_y^{a_x(y)}, W^{-c})$. This means that

$$S(y) \leq a_x(y)$$

then,

$$S(y) \leq a_x(y) = S(x) + \varepsilon + C|x - y|$$

for the arbitrarity of $\varepsilon > 0$,

$$S(y) \leq S(x) + C|x - y| : \qquad S(y) - S(x) \leq C|x - y|$$

By interchanging the role of x and y, we finally obtain

$$|S(y) - S(x)| \leq C|x - y|, \qquad \forall x, y \in U$$

$\square$

• In other words: $S(x)$ inherits the **same** Lip constant $C > 0$ from $W(x, u)$.

[5] $W_x^{-c} = W^{-c}$, $\forall x \in U$ and c large

Variational solutions are not 'Markovian'

• There is a 'drawback' of the variational solution $C^{0,1}$: it is inherited from a generating function of a Lagrangian submanifold, starting from a **smooth**, C^1, initial function $\sigma : N \to \mathbb{R}$,
Consider the application J:

$$J : C^{1,1}([0,T] \times T^*N) \times C^1(N) \to C^{0,1}([0,T] \times N)$$

$$(H, \sigma) \mapsto u =: J(H,\sigma)(t) = S(t,q)$$

Theorem *The application J is uniformly continuous if all the spaces are equipped with the C^0 topology. Thus it extends to an uniformly continuous map, still denoted by J,*

$$J : C^{0,1}([0,T] \times T^*N) \times C^0(N) \to C^0([0,T] \times N)$$

in particular, fixed H,

$$\|J(H,\sigma_1) - J(H,\sigma_2)\|_{C^0} \leq \|\sigma_1 - \sigma_2\|_{C^0}.$$

• Note: it is the same **non-expansive** property of the Lax-Oleinik semi-group !

Variational solutions are not 'Markovian'

Every continuous $\sigma \in C^0(N)$ can be approximated in the uniform convergence by a sequence of differentiable $\sigma_n \in C^1(N)$,
The related variational solution is $J(H, \sigma_n) = u_{\sigma_n}$.
By continuity of J,
$i)$ u_{σ_n} is a Cauchy sequence and
$ii)$ its limit is independent of the approximating sequence σ_n.
$\Rightarrow$:
Definition: C^0-variational solution *Given a continuous initial datum $\sigma \in C^0(N)$, the C^0-variational solution for the Cauchy problem is the unique function $u_\sigma \in C^0([0,T] \times N)$ such that, for any arbitrary C^1 approximating sequence σ_n:*

$$C^1(N) \ni \sigma_n \xrightarrow{C^0} \sigma \in C^0(N),$$

with related $C^{0,1}$-variational solutions $J(H, \sigma_n) = u_{\sigma_n}$, we have that

$$\lim_{n \to +\infty} \|u_{\sigma_n} - u_\sigma\|_{C^0} = 0 \qquad \textit{on } [0,T] \times N. \tag{3}$$

A topological algebraic framework

- Main Theorem: *Let N be compact, and*

$$S : N \times \mathbb{R}^n \ni (x, \xi) \longmapsto S(x, \xi) \in \mathbb{R}$$

be a GFQI. Then, up to a shift of the degree by k_-:

$$\boxed{H^*(S^{\infty}, S^{-\infty}) \cong H^*(N)}$$

Meaning of the Main Theorem:

$$H^*(S^\infty, S^{-\infty}) \cong H^*(N)$$

$\Rightarrow$ For compact N, the **absolute** cohomology $H^*(N)$,
is precisely the **relative** cohomology
of the sublevel sets of generic functions on $f : N \to \mathbb{R}$:

$$\text{for} \quad c > 0 : \qquad -c < \min f \leq \max f < c,$$

$$H^*(f^\infty, f^{-\infty}) = H^*(f^c, f^{-c}) = H^*(N, \emptyset) \cong H^*(N)$$

In other words:
To look for critical values and critical points of GFQI $S : N \times \mathbb{R}^k \to \mathbb{R}$

is like

looking for critical values and critical points of $f : N \to \mathbb{R}$!

•

Proof. Since S is a GFQI, for $c > 0$ big enough,

$$S^{\pm c} \cong N \times Q^{\pm c} =: S^{\pm\infty}$$

It follows $\Big($remembering Kunneth formula:

$H^n(M \times Q^c, M \times Q^{-c}) \simeq \bigoplus_{p+q=n} H^p(M) \otimes H^q(Q^c, Q^{-c})\Big)$:

$$H^*(S^\infty, S^{-\infty}) \cong H^*(N \times Q^\infty, N \times Q^{-\infty}) \cong H^*(N) \otimes H^*(Q^\infty, Q^{-\infty}) \cong$$

$$\cong H^*(N) \otimes H^*(D^{k_-}, \partial D^{k_-}) \cong H^*(N) \otimes H^*_c(\mathbb{R}^{k_-}) \cong H^*_c(N \times \mathbb{R}^{k_-}) \cong H^*(N)$$

where the last one is realized by the **Thom isomorphism**: giving the negative bundle

$$\pi : N \times \mathbb{R}^{k_-} \longrightarrow N$$

and denoting by t^{k_-} the **Poincaré dual cohomological class** of the null section ($=N$) of π, we get the k_--shifted isomorphism:

$$H^h(N) \ni \alpha \longmapsto T(\alpha) := \pi^*\alpha \wedge t^{k_-} \in H^{h+k_-}_c(N \times \mathbb{R}^{k_-})$$

□

A **metric** on the Lagrangian submanifolds set $\mathcal{L}$

- We denoted by $c(\alpha, L)$ the min-max critical value of a GFQI relative to a Lagrangian submanifold $L \in \mathcal{L}$
- $\mathcal{L}$: the set of Lagrangian submanifolds of T^*N which are Hamiltonian isotopic to $\mathcal{O}_{T^*N}$
- $L_1, L_2 \in \mathcal{L}$ be generated by the GFQI $S_1(x;\xi)$ and $S_2(x;\eta)$ respectively.
- We denote by $(S_1 \sharp S_2)(x;\xi,\eta)$ the GFQI

$$(S_1 \sharp S_2)(x;\xi,\eta) := S_1(x;\xi) + S_2(x;\eta)$$

- Considering

$$(S_1 \sharp (-S_2))(x;\xi,\eta) = S_1(x;\xi) - S_2(x;\eta)$$

we note that its **critical points** of $(S_1 \sharp (-S_2))$ are precisely marking the **intersections** $L_1 \bigcap L_2$:

$$(x, p_1) \in L_1,\ \ (x, p_2) \in L_2 : \quad 0 = \frac{\partial S_1}{\partial x} - \frac{\partial S_2}{\partial x} = p_1 - p_2, \quad 0 = \frac{\partial S_1}{\partial \xi}, \quad 0 = \frac{\partial S_2}{\partial \eta}$$

- The main theorem, here in the form

$$H^*\Big((S_1 \sharp (-S_2))^{\infty}, (S_1 \sharp (-S_2))^{-\infty}\Big) \cong H^*(N)$$

is telling us that we have simply to look at (the cohomology of) the base manifold N in order to find global critical points of $S_1 \sharp (-S_2)$.
This leads us to the

- Definition:

$$\gamma(L_1, L_2) := c\,(\mu, S_1 \sharp (-S_2)) - c\,(1, S_1 \sharp (-S_2))\,,$$

where $1 \in H^0(N)$ and $\mu \in H^n(N)$ are generators.

- $\gamma(L_1, L_2)$ is a **metric** on $\mathcal{L}$.

•
Definition Let $(\phi_t)_{t\in[0,1]}$ be a Hamiltonian isotopy, $\phi=\phi_1$. We set

$$\tilde{\gamma}(\phi) := \sup_{L\in\mathcal{L}} \gamma(\phi(L), L)$$

All the Hamiltonians are now assumed to be compactly supported

• **Proposition**

1. $\tilde{\gamma}(\phi)\geq 0$ and $\tilde{\gamma}(\phi)=0$ if and only if $\phi=$ id,
2. $\tilde{\gamma}(\phi)=\tilde{\gamma}(\phi^{-1})$,
3. $\tilde{\gamma}(\phi\circ\psi)\leq\tilde{\gamma}(\phi)+\tilde{\gamma}(\psi)$ (triangle inequality),
4. $\tilde{\gamma}(\psi\circ\phi\circ\psi^{-1})=\tilde{\gamma}(\phi)$ (invariance by conjugation).

In particular,

$$d(\phi_1,\phi_2) := \tilde{\gamma}(\phi_2^{-1}\circ\phi_1)$$

defines a **metric** on the group of Hamiltonian diffeomorphisms of T^*N.

• **Proposition** *Assume that ϕ is the time-one map associated to the Hamiltonian $H(t,x,p)$. Then we have*

$$\tilde{\gamma}(\phi)\leq\|H\|_{C^0}$$

where: $\|H\|_{C^0} := \sup_{[0,T]\times T^*N} H(t,x,p) - \inf_{[0,T]\times T^*N} H(t,x,p)$

Consequences on variational solutions of H-J

- Beside to the above

$$\boxed{\tilde{\gamma}(\phi) \leq \|H\|_{C^0}}$$

we have also:

- **Proposition** Let L_1, L_2 and u_1, u_2 be the **geometric** and **variational solutions** for the Cauchy problems of **H-J** referred to the initial data σ_1 and σ_2 respectively. Then we have

$$\boxed{\|u_1 - u_2\|_{C^0} \leq \gamma(L_1, L_2)}$$

- At the end, we need **both** them to gain:

$$\|J(H, \sigma_1) - J(H, \sigma_2)\|_{C^0} \leq \|\sigma_1 - \sigma_2\|_{C^0}$$

- **Poincaré last geometrical theorem** Take a Hamiltonian on the *cylinder* $T^*\mathbb{T}^1$ like:

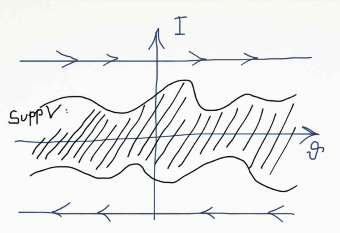
$$H(\theta, I) = \frac{|I|^2}{2} + V(\theta, I)$$

V is compactly supported on $T^*\mathbb{T}^1$, for $|I| > K$: $V \equiv 0$

$$\text{for} \quad I < -K \quad : \quad \dot{I} = 0, \qquad \dot{\theta} = I < 0,$$

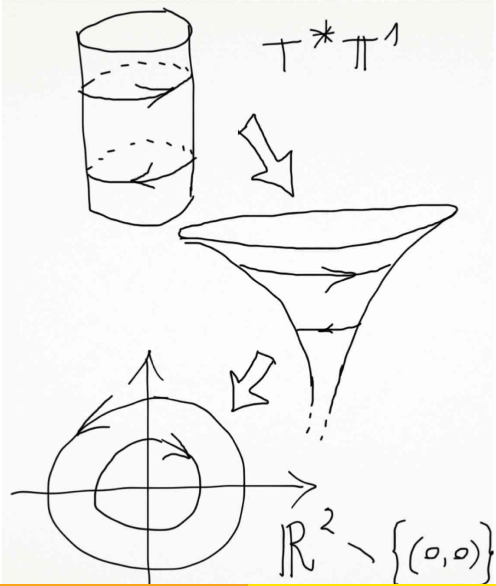
$$\text{for} \quad I > K \quad : \quad \dot{I} = 0, \qquad \dot{\theta} = I > 0,$$

- consider $I^* > K$, and the time τ-flow Φ_H^τ for τ : $I^*\tau < 2\pi$ on the 'strip' between I^* and $-I^*$:



- so we restore a *twist-like condition* of the Poincaré last geometrical theorem....

- ...we should 'open' symplectically the cylinder $T^*\mathbb{T}^1$ over $\mathbb{R}^2 \setminus \{(0,0)\}$:



- The symplectic twist map (Can. Transf.) of the annulus $\mathcal{A}$ into itself

$$\Phi_H^\tau : \mathcal{A} \longrightarrow \mathcal{A}$$

$$(\theta_0, I_0) \longmapsto (\theta_1, I_1)$$

admits a Generating Function Quadratic at Infinity, $F(\theta_0, \theta_1; \xi)$:

$$I_0 = -\frac{\partial F}{\partial \theta_0}(\theta_0, \theta_1; \xi), \qquad I_1 = \frac{\partial F}{\partial \theta_1}(\theta_0, \theta_1; \xi) \qquad 0 = \frac{\partial F}{\partial \xi}(\theta_0, \theta_1; \xi)$$

- In order to find fixed points of Φ_H^τ,

(i) we consider the composition of F with the **diagonal**:

$$S(\theta; \xi) := F(\theta, \theta; \xi) \qquad \text{(is, again, a GFQI)}$$

(ii) we search the global critical points of S, i.e., *both* respect to θ *and* respect to ξ:

Some other consequences of the topological algebraic framework

- the crit. points of S **are** the fixed points of Φ_H^τ
- Finally, the above main theorem, here for $N = \mathbb{T}^1$,

$$H^*(S^\infty, S^{-\infty}) \cong H^*(\mathbb{T}^1)$$

tells us that we have to look at the cohomology of the torus $\mathbb{T}^1$, precisely

$$\underbrace{\#\{fixed\ point\ of\ \Phi_H^\tau\} \geq cl(\mathbb{T}^1)}_{lower\ bound\ of\ Lusternik-Schnirelman} = 2 \qquad cl: \text{ cup-length} \approx \text{'category'}$$

- The above result can be thought in any dimension n:

$$\#\{fixed\ point\ of\ \Phi_H^\tau\} \geq cl(\mathbb{T}^n) = n + 1$$

- towards Arnol'd conjecture....

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THE END

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